replaced the circular cylinder by an equilateral triangular one, with one of its flat surfaces facing upstream and an edge pointing downstream. Then the spanwise crossflow is found to disappear; the streamwise acceleration around the two sharp edges of the triangle apparently washes the vortices downstream too rapidly to allow the formation of the secondary spanwise transport.

With regard to the fluid-dynamical details, cross-transport through the vortex core appears to stem from two sources: 1) the solenoidal property of vorticity, and 2) the no-slip condition on a solid surface. (For a demonstration of this, see Ref. 1.) It is indeed from the latter condition, by which the normal component of vorticity is reduced to zero, in combination with the former property that excludes the possibility of the vortex tubes ending on any stationary solid surface.[4] This results in the enlargement, toward the surface, of the cross-sectional area of the vortex tube; the area change, together with the no-slip condition again, causes the variation of the swirl velocity in the direction of the vortex axis. This change in the circumferential velocity sets up, in turn, the pressure gradient along the vortex core, which then induces the cross transport. In addition to the preceding mechanism, which is applicable even to a stationary fluid, the sheared velocity profile within the boundary layer over the sidewalls may further enhance the foregoing effect, in much the same manner as discussed in Ref. 5.

Acknowledgment

This work was supported by the Air Force Office of Scientific Research under Grant No. AFOSR-83-0049 to the University of Tennessee Space Institute and AFSOR-87-0258 DEF to the University of Washington. Thanks are due to G. Yu (presently at Beijing Institute of Mechanics) for taking the smoke photograph.

References

[1]Koochesfahani, M., "Vortical Patterns In the Wake of an Oscillating Airfoil," AIAA Paper 87-0111, 1987.

[2]Roberts, F. A. and Roshko, A., "Effects of Periodic Forcing on Mixing in Turbulent Shear Layers and Wakes," AIAA Paper 85-0570, 1985.

[3]Kurosaka, M., Gertz, J. B., Graham, J. E., Goodman, J. R., Sundaram, P., Riner, W. C., Kuroda, H., and Hankey, W. L., "Energy Separation in a Vortex Street," *Journal of Fluid Mechanics*, Vol. 178, May 1987, pp. 1-29.

[4]Lighthill, M. J., "Introduction: Real and Ideal Fluids," *Laminar Boundary Layers*, edited by L. Rosenhead, Oxford Univ. Press, London, 1963, Chap. 1, p. 51.

[5]Kurosaka, M. Marble, F. E., Goodman, J. R., Wohlman, R. A., and Tirres, L. "Spontaneous Decrease of Local Entropies Observed in a Shear Flow" (in preparation).

Interpretation of Jet Mixing Using Fractals

F. C. Gouldin*
Cornell University, Ithaca, New York

Introduction

IN Ref. 1, Mandelbrot proposes that constant scalar property surfaces in homogeneous, isotropic turbulence are fractal surfaces and suggests two values for the fractal dimension D, depending on the spatial variance of the scalar field: $D = 2\frac{1}{2}$ for Gauss-Burger turbulence and $D = 2\frac{2}{3}$ for Gauss-Kolmogorov turbulence. Lovejoy[2] for clouds and Sreenivasan and Meneveau[3] for shear flows obtain data suggesting fractal dimensions between approximately 2.32 and 2.4 for constant-property surfaces. Hentschel and Procaccia[4] and Kingdon and Ball[5] report analyses of cloud dispersion in homogeneous, isotopic turbulence. In these analyses, molecular transport is ignored, high Reynolds number is assumed, and the dispersion of one species into another is modeled. The structure function $S(l,t)$ for a scalar θ, defined over a volume V as

$$S(l,t) = \frac{1}{V}\int \mathrm{d}r\langle|\theta(r + l,t) - \theta(r,t)|^2\rangle$$

is introduced in the modeling. Both Hentschel and Procaccia and Kingdon and Ball suggest a form for $S(l,t)$ as $|l|\rightarrow 0$. Namely, $\lim_{|l|\rightarrow 0} S(|l|) \sim |l|^B$. Both analyses contain heuristic elements, and there is a fundamental difference between the two sets of authors in their interpretation of B and its relationship to D. Yet their final expressions for D are the same:

$$D = 2 + \frac{2+\mu}{6} \tag{1}$$

where μ is the intermittency exponent.

In this note, a semi-empirical model of mixing in jet flows is developed that assumes the instantaneous jet structure can be represented by an ensemble of fractal, constant-property surfaces. The model gives an expression for the mean jet fluid concentration on the jet centerline and, for Reynolds number independence, requires that $D = 2 + [(2+\mu)/6]$. This expression for D is the major finding to be reported. While it is identical to that of others, it is obtained by very different reasoning and for a different flow, a free shear flow.

Jet Mixing Model

For analysis, the following major assumptions are made.

1) The flow is an axisymmetric jet issuing into a stagnant fluid with uniform and constant density.

2) The turbulence is stationary, and the flow is Reynolds number independent.

3) The distributions of mean composition and mean velocity follow similarity.

4) Surfaces of constant, instantaneous jet fluid mass fraction may be represented by fractal surfaces for the purpose of estimating the ensemble mean area of such surfaces.

The instantaneous jet structure can be viewed as a set of constant jet fluid mixture fraction Z surfaces, and it is hypothesized that these surfaces exhibit fractal character over a range of length scales associated with the scales of the turbulent velocity fluctuations, i.e., from the Kolmogorov scale η to the integral scale l. For a constant Z surface of area S enclosing a volume V, an integral equation for jet fluid mass conservation can be written. Then, for stationary turbulence, one can obtain, by taking the ensemble average of this equation, an equation expressing a balance between the mean flux of jet fluid issuing from the jet M_j and the mean jet fluid mass flux across the Z constant surface:

$$M_j = \pi(\mathrm{d}/2)^2\rho V_j = \langle \int_S [\rho Z(\boldsymbol{q} - \boldsymbol{q}_b) + \boldsymbol{j}_z]\cdot \mathrm{d}\boldsymbol{s}\rangle \tag{2}$$

where V_j is the initial volume flux weighted average jet velocity, d is the initial jet diameter, $\boldsymbol{q}$ is the fluid velocity, $\boldsymbol{q}_b$ is the Z surface velocity, and $\boldsymbol{j}_z$ is the diffusive flux of jet fluid across the Z surface. It is assumed that the Z surface is attached to the lip of the jet and that this surface is simply connected. The first assumption is valid, and if the second assumption is relaxed, it seems reasonable to argue that the average flux across the Z surface can still be given by the right-hand side of Eq. (2).

Received May 22, 1987; revision received Jan. 18, 1988. Copyright © American Institute of Aeronautics and Astronautics, Inc., 1988. All rights reserved.

*Professor of Mechanical and Aerospace Engineering. Associate Fellow AIAA.

Consider the segment of a Z surface between the axial points x and $x + dx$. The contribution from the Z surface in this region to the right-hand side of Eq. (2) is modeled as the product of the mean flux of jet fluid across the surface per unit area times the mean surface area dA_Z between x and dx.

Represent the Z surface geometry by a fractal surface with fractal dimension D, a minimum scale of surface wrinkling, the inner cutoff (ε_i), and a maximum scale of wrinkling, the outer cutoff (ε_o). For a fractal surface with scales of wrinkling between ε_i and ε_o, the measured surface area A varies with measurement scale ε, as shown in Fig. 1. Fractal behavior is given by the power-law dependency of A upon ε. For $\varepsilon \to 0$, if the surface has a minimum scale of wrinkling, A must approach a constant value A_i, the true value (see the horizontal line in Fig. 1 at $A = A_i$), and ε_i is defined quantitatively by $A_i/L^3 \sim (\varepsilon_i/L)^{2-D}/L$. For a Z surface, molecular diffusion is expected to limit the minimum scale of wrinkling, and it is assumed that $\varepsilon_i = \eta$. (Since the ratio A_i/A_o is required, the constant of proportionality is unimportant in this application.)

For $\varepsilon > \varepsilon_o$, the measured area is also a constant, $A = A_o$ (see Fig. 1), and it is assumed that $\varepsilon_o = l$. This behavior for large ε implies that the surface appears smooth and unwrinkled at large measurement scale, which, for this application, requires that ε_o be much less than the radius of curvature of the $\langle Z \rangle$ surface. Such limiting behavior for large ε is reasonable for shear flows but may not be reasonable for homogeneous, isotopic turbulence.

In Ref. 6, it is argued that A_i is a good estimate of the ensemble mean surface area of a constant property surface, i.e., between x and $x + dx$ $dA_Z = A_i$. It follows from Fig. 2 and the assumptions $\varepsilon_i = \eta$ and $\varepsilon_o = l$ that

$$A_i/A_o = (\varepsilon_o/\varepsilon_i)^{D-2} = (l/\eta)^{D-2} \tag{3}$$

The choice of η as the inner cutoff implies a unit Schmidt number. For homogeneous, isotopic turbulence, $l/\eta = A_t^{1/4} R_l^{3/4}$,[6] with A_t a constant and R_l the turbulence Reynolds number expressed in terms of the absolute turbulence intensity and the integral scale. Since the Z surface is imbedded in fully turbulent fluid, it seems reasonable to use this expression in the current analysis, and therefore

$$A_i/A_o = (A_t^{1/4} R_l^{3/4})^{D-2} \tag{4}$$

As noted, A_o is the surface area measured for $\varepsilon > \varepsilon_o = l$. It is assumed that A_o for a Z surface is equal to the area of the $\langle Z \rangle$ constant surface for which $\langle Z \rangle$ equals Z. Implicit in this assumption is the idea that the Z surface is fluctuating about the position of the $\langle Z \rangle$ surface. To help make this point, a sketch of the intersections of a $\langle Z \rangle$ constant surface and of realizations of a Z constant surface with a plane containing the jet axis are depicted in Fig. 2. The area of the $\langle Z \rangle$ surface in the region x to $x + dx$ is $dA_{\langle Z \rangle} = 2\pi r_{\langle Z \rangle} dx$, and with $dA_Z/dA_{\langle Z \rangle} = A_i/A_o$, one obtains

$$dA_Z = 2\pi r_{\langle Z \rangle} (A_t^{1/4} R_l^{3/4})^{D-2}\, dx \tag{5}$$

where $r_{\langle Z \rangle}$ is the radial location of the $\langle Z \rangle$ surface at x.

The flux of jet fluid across a Z surface is the result of convection $(\boldsymbol{q} - \boldsymbol{q}_b)$ and diffusion (j_z). To estimate the mean of these two quantities, a flux velocity given by $u_d = D/\eta_d$ is introduced, where η_d is an effective length scale defined by the preceding expression. It follows that the total mean mass flux across a Z surface per unit area is $\rho Z u_d$. (Here, D is the diffusion coefficient. There should be no confusion with the fractal dimension, since the latter appears as an exponent.) Gibson[7] has considered the motion of a constant scalar property surface in turbulent flow, and from his analysis,

$$(\boldsymbol{q} - \boldsymbol{q}_b) = -D(\nabla^2 Z/|\nabla Z|)(\nabla Z/|\nabla Z|)$$

For Fick's law diffusion

$$j_z = -\rho Z D \nabla \ln Z$$

Thus, the appropriate length scale for u_d is related to

$$\langle |(\nabla^2 Z/|\nabla Z|)(\nabla Z/|\nabla Z|) + \nabla \ln Z| \rangle^{-1}$$

One expects intuitively that the largest contributions to η_d come from regions in the flow where Z is varying rapidly with distance, and, hence, both $\nabla^2 Z$ and $\nabla \ln Z$ are large. These regions contain the smallest turbulent eddies, and an appropriate estimate for η_d would seem to be a quantity proportional to η for a unit Schmidt number. Thus, η_d is replaced by $(1/C_d)\eta$, and $u_d = C_d v$, where v is the Kolmogorov velocity scale and C_d is a model constant.

For homogeneous, isotopic turbulence, $v = A_t^{1/4} R_l^{-1/4} u'$. As argued, the Z surface is embedded fully in turbulent fluid, and therefore it seems reasonable to use this expression to relate v to the absolute turbulent intensity u' and to R_l. In addition, it is assumed that u' can be expressed in terms of the jet centerline mean velocity u_0: $u' = (u'/u_0)u_0$, with u'/u_0 being constant over the jet. Conditional velocity measurements in jets show that u'/u_0 does not vary greatly. By substitution,

$$u_d = C_d A_t^{1/4} R_l^{-1/4} (u'/u_0) u_0 \tag{6}$$

According to Gibson's expression, $\boldsymbol{q} - \boldsymbol{q}_b$ is undefined or infinite at critical points, i.e., at points where $\nabla Z = 0$, and one

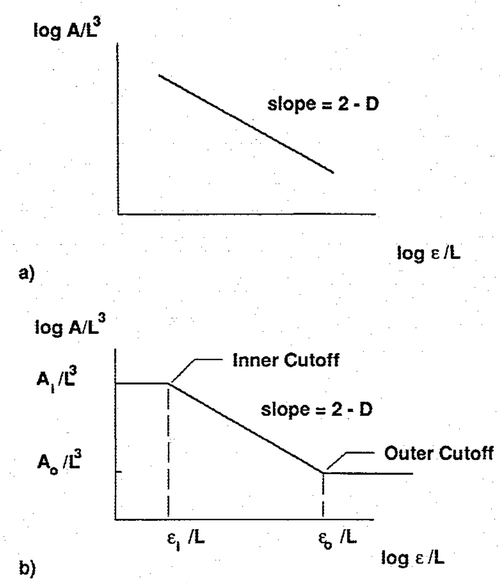

Fig. 1 Variation with measurement scale of measured area in a cubic volume of side L for a surface exhibiting fractal behavior: a) no inner or outer cutoffs; b) inner and outer cutoffs to fractal behavior. From Mandelbrot's line of argument[1] for measurement scales in the fractal range, $A/L^3 \sim (\varepsilon/L)^{-D}\varepsilon^2/L^3$.[7]

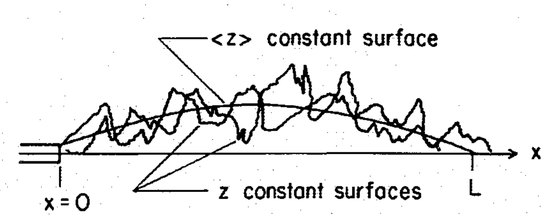

Fig. 2 Sketch depicting the intersection of $\langle Z \rangle$ and Z constant surfaces with a plane containing the jet axis. The curves need not be continuous. The curves associated with Z surfaces should have D values one less than the corresponding surface.

may question whether the integral in Eq. (2) converges. On physical grounds, one expects $q - q_b$ at critical points to be infinite and to be large in the region of a critical point. Thus, it is argued that Z surfaces will move away from critical points very rapidly, and hence the effect of such points on the integral are negligible. Alternatively, one can argue, again on physical grounds, that the ensemble average of the integral of $(q - q_b) \cdot ds$ over the area of a constant-property surface must be finite even if $q - q_b$ at singular points is not. Then $\langle q - q_b \rangle$ can be defined from the ensemble average mass flux integral.

By combining Eqs. (2), (5), and (6) and integrating along the x axis, one obtains

$$M_j = \rho Z \int_0^L C_d A_i^{1/4} R_l^{-1/4} (u'/u_0) u_0 2\pi r_{\langle Z \rangle} (A_i^{1/4} R_l^{3/4})^{D-2} \, dx \tag{7}$$

To proceed further, relationships for $r_{\langle Z \rangle}$ and u_o as functions of x are required, and empirical, similarity-form expressions are used. Having taken such a step, one cannot claim that the resulting expression for centerline concentration is a predictive one. The exercise does, however, allow one to determine whether or not the fractal picture gives results consistent with experiment, thereby providing a test of the applicability of fractal concepts.

Using standard similarity relationships[8] and an exponential variation of $\langle Z \rangle$ with r[9], one can show, following integration, that

$$V_j = (\pi/2)^{1/2} (L_z/C_u a) C_d A_i^{(D-1)/4} R_l^{(3/4(D-2)-1/4)} (u'/u_0) U_j Z(L/d) \tag{8}$$

where L_z, C_u, and a are constants appearing in the similarity expressions.[8,9] Momentum and volume average velocities are U_j and V_j, and their ratio will vary with initial conditions; let $V_j/U_j = b$. The axial distance at which the mean jet fluid mass fraction on the centerline equals Z is L.

For similarity, Eq. (8) must be Reynolds number independent, and therefore $h(D) = (3/4(D-2) - 1/4) = 0$. With this condition satisfied, Eq. (8) is consistent with the expected similarity form and is quantitatively equivalent, provided that

$$(\pi/2)^{1/2} (L_z/C_u ab) C_d A_i^{(D-1)/4} (u' u_0) = C_z$$

where C_z is a constant in the expression for the axial decay of the centerline value of $\langle Z \rangle$. Substitution with reasonable values of the constants[8,9] into the latter expression gives $C_d \approx 0.38$, which is a quite reasonable result.

The analysis to this point has not considered the intermittency of viscous dissipation. With intermittency, the dissipative length scale l_d replaces η as the inner cutoff, and from Frisch et al.,[10] $l_d \sim l R_l^{3/(4-\mu)}$, u_d is scaled as $E_d^{1/2}$, where E_d is "the kinetic energy per unit mass on scales" $\sim l_d$.[10] It can be shown that this scaling gives $u_d = u' R_l^{1/2 - 3/(4-\mu)}$. Thus, with intermittency, the Reynolds number dependency in Eq. (8) becomes $R_l^{[3/(4-\mu)][D-2]} R_l^{1/2 - 3/(4-\mu)}$, and for Reynolds number independence, $D = 2 + (\mu + 2)/6$.

This result is identical to that obtained by Hentschel and Procaccia[4] and by Kingdon and Ball,[5] while for $\mu = 0$, it is not equivalent to Mandelbrot's result ($D = 2\,2/3$) for Gauss-Kolmogorov turbulence,[1] even though in both cases the variance scales as separation distance to the 5/3 power. In his development, Mandelbrot assumes a Gaussian random field; no such assumption is made in the present development. It is noted again that the approach taken by Hentschel and Procaccia and by Kingdon and Ball is quite different than the present one. It is especially noteworthy that the present analysis is for a free shear flow rather than for homogeneous, isotropic turbulence. However, expressions for l/η and v valid for homogeneous, isotropic turbulence are employed in the modeling. Their use is justified by noting that Z surfaces are embedded fully in turbulent fluid where conditions should approach those in homogeneous, isotropic turbulence.

Summary and Closing Comments

A fractal representation of constant Z surfaces in axisymmetric jets is used with similarity and empirical results to obtain an expression for the centerline mean jet fluid mixture that is of appropriate form and numerically correct if a reasonable value for C_d, the only new empirical constant in the modeling, is chosen. The result provides support for the hypothesis that constant-property surfaces in free shear flows are fractal-like.

For Reynolds number independence, the R_l term occurring in the surface area expression must cancel the R_l term appearing in the expression for u_d, which requires that $D = 2 + (\mu + 2)/6$. This expression for D is identical to that obtained in Refs. 4 and 5. The observed range of values for μ is $0.25 \le \mu \le 0.5$[4], and the corresponding range for D is $2.33 < D < 2.42$, which compares favorably with experiments.[2,3]

The fractal representation is used to obtain an expression for mean surface area, Eq. (5). To obtain Eq. (5), it is necessary to identify an outer cutoff area and to assume that this area is equal to the $\langle Z \rangle$ surface area. This step is justified a posteriori by the success of the modeling. It is worthy of note that the expression for D can be obtained without identifying an expression for the outer cutoff area. Equation (1) is contingent on several steps: 1) expressing dA_Z in terms of A_i/A_o, 2) equating A_i/A_o to $R_l^{[3/(4-\mu)][D-2]}$, 3) scaling u_d by $u' R_l^{1/2 - 3/(4-\mu)}$, with u' obtained from similarity, and 4) requiring Reynolds number independence. Steps 1) and 2) form the essential features of the fractal hypothesis, while step 4) is supported by experiment. Of the four steps, step 3) is perhaps the most questionable.

Acknowledgments

Support from the SERC and from the U.S. Army Research Office (Contract DAAG29-82-K 0187) is acknowledged. This work was initiated while the author was a Science and Engineering Research Council Visiting Fellow at the University Engineering Department, Cambridge, England. Helpful discussions with K. N. C. Bray of Cambridge and with S. B. Pope of Cornell University are gratefully acknowledged.

References

[1]Mandelbrot, B. B., "On the Geometry of Homogeneous Turbulence with Stress on the Fractal Dimension of the Iso-Surfaces of Scalars," *Journal of Fluid Mechanics*, Vol. 72, Dec. 1975, pp. 401–416.

[2]Lovejoy, S., "Area-Perimeter Relationships for Rain and Cloud Areas," *Science*, Vol. 216, April 1982, pp. 185–187.

[3]Sreenivasan, K. R. and Meneveau, C., "The Fractal Facets of Turbulence" *Journal of Fluid Mechanics*, Vol. 173, Dec. 1986, pp. 356–386.

[4]Hentschel, H. G. E. and Procaccia, I., "Relative Diffusion in Turbulent Media: The Fractal Dimension of Clouds," *Physical Review A—General Physics*, Vol. 29, March 1984, pp. 1461–1470.

[5]Kingdon, R. D. and Ball, R. C., "The Fractal Dimension of Clouds," *Journal of Physics A: General Physics* (submitted for publication).

[6]Gouldin, F. C., "An Application of Fractals to Modeling Premixed Turbulent Flames," *Combustion and Flame*, Vol. 68, June 1987, pp. 249–266.

[7]Gibson, C. H., "Fine Structure of Scalar Fields Mixed by Turbulence: I. Zero-Gradient Points and Minimal Gradient Surfaces," *Physics of Fluids*, Vol. 11, Nov. 1968, pp. 2305–2315.

[8]Gouldin, F. C., Schefer, R. W., Johnston, S. C., and Kollman, W., "Nonreacting Turbulent Mixing Flows," *Progress in Energy and Combustion Science*, Vol. 12, Dec. 1986, pp. 257–303.

[9]Pitts, W. M. and Kasawagi, T., "The Application of Laser-Induced Rayleigh Light Scattering to the Study of Turbulent Mixing," *Journal of Fluid Mechanics*, Vol. 141, April 1984, pp. 391–429.

[10]Frisch, U., Sulem, P.-L. and Nelkin, M., "A Simple Dynamical Model of Intermittent Fully Developed Turbulence," *Journal of Fluid Mechanics*, Vol. 87, Aug. 1978, pp. 719–736.